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# On the Existence of Solutions of the Cauchy Problem for a Nonlinear Diffusion Equation

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## 1. Introduction

We investigate the Cauchy problem for the following nonlinear diffusion equation,

$$(1.1) \qquad \frac{\partial}{\partial t}(|u|^{\beta-1}u) = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \qquad \text{in} \quad S_T, \quad \beta > 0,\ p > 1,$$

$$(1.2) \qquad |u|^{\beta-1}u(\cdot, 0) = \mu(\cdot) \qquad \text{in} \quad \mathbf{R}^N.$$

Here $S_T = \mathbf{R}^N \times (0, T)$, $0 < T < \infty$, and $\mu$ is an $L^1_{\mathrm{loc}}(\mathbf{R}^N)$-function or a $\sigma$-finite Borel measure.

The equation (1.1) is called the doubly nonlinear parabolic equation, which contains the heat equation (*i.e.* $\beta = 1, p = 2$), the porous medium equation (*i.e.* $\beta > 0, p = 2$), and the $p$-Laplacian equation (*i.e.* $\beta = 1, p > 1$), and the equation (1.1) has been studied by several authors, for example, see [**7**], [**8**], [**10**], [**11**], and [**13**]. We distinguish the Cauchy problem (1.1) with (1.2) into three cases:

$$\text{(I)} \quad (p-1)/\beta > 1, \qquad \text{(II)} \quad (p-1)/\beta = 1, \qquad \text{(III)} \quad 0 < (p-1)/\beta < 1,$$

and study the existence of the solution, respectively. For each case of (I), (II) and (III), the behavior of solutions of (1.1) is completely different from one another, and so we need this separation. The case of (I) contains the so-called degenerate case of porous medium equation and $p$-Laplacian equation, and the case of (III) contains the singular case of them. In what follows, we call (I) the degenerate case and (III) the singular case, respectively.

A classical result of A.N.Tychonov [**12**] states that the Cauchy problem for the heat equation, $u_t = \Delta u$, has a unique classical solution in the strip $S_T$ for continuous initial data $\mu(x)$ satisfying

$$(1.3) \qquad |\mu(x)| \le C \exp(|x|^2/4T) \qquad \text{as} \quad |x| \to \infty.$$

Moreover D.G.Aronson [1] generalized them for parabolic operator with variable coefficients:

$$\mathcal{L}u \equiv \frac{\partial}{\partial t}u - \frac{\partial}{\partial x_j}\left\{A_{ij}(x,t)\frac{\partial}{\partial x_i}u + A_j(x,t)u\right\} \tag{1.4}$$

with suitable conditions imposed on coefficients $A_{ij}(x,t)$ and $A_j(x,t)$. For the Cauchy problem for the equation $\mathcal{L}u = 0$ with the initial data $\mu(x)$ satisfying

$$\int_{\mathbf{R}^N} \mu(x)\exp(-\Lambda|x|^2)dx < \infty, \qquad \Lambda > 0, \tag{1.5}$$

he proves that it has a unique classical solution in some strip $S_{T'}$, where $T'$ is a constant dependent on $\Lambda$. Furthermore he proved the solution $u$ is written by the form in the strip $S_{T'}$

$$u(x,t) = \int_{\mathbf{R}^N} \Gamma(x,t;\xi,0)u_0(\xi)d\xi,$$

where $\Gamma$ is the fundamental solution of $\mathcal{L}u = 0$. See also [9] and [14].

For the degenerate case of porous medium equation

$$u_t = \Delta(u^m), \quad m > 1, \tag{1.6}$$

Ph.Benilan, M.G.Crandall and M.Pierre [2] proved that the Cauchy problem is uniquely solvable in the sense of weak solutions for the initial data satisfying

$$\limsup_{\rho\to\infty} \rho^{-N-2/(m-1)} \int_{B_\rho} d|\mu| < \infty, \tag{1.7}$$

where $B_\rho$ is a ball of radius $\rho > 0$ with center 0. On the other hand, for the degenerate case of the $p$-Laplacian equation,

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad p > 2, \tag{1.8}$$

E.DiBenedetto and M.A.Herrero [4] proved the similar result for the initial data satisfying

$$\limsup_{\rho\to\infty} \rho^{-N-p/(p-2)} \int_{B_\rho} d|\mu| < \infty. \tag{1.9}$$

Furthermore E.DiBenedetto and M.A.Herrero [5] and E.DiBenedetto and T.C.Kwong [6] studied the Cauchy problem for singular cases of the porous medium equation ($(N-2)^+/2 < m < 1$) and the $p$-Laplacian equation ($2N/(N+1) < p < 2$), and obtained the $L^\infty_{\text{loc}}(\mathbf{R}^N)$-estimate of the solution for the $L^1_{\text{loc}}(\mathbf{R}^N)$ initial data.

Our purpose of this paper is to extend the earlier results on the existence of solutions to the doubly nonlinear parabolic equation (1.1). The main point of this paper is to treat the case (II).

For the case of (II), if the initial data $\mu$ satisfies

$$\int_{\mathbf{R}^N} \exp(-\Lambda|x|^{p/(p-1)})d|\mu| < \infty \tag{1.10}$$

for some constant $\Lambda > 0$, then we prove that there exists a weak solution of (1.1) with (1.2) in the strip $S_{T(\Lambda)}$, where $T(\Lambda) = (p-1)^{2(p-1)}\Lambda^{1-p}/p^p$. Here we remark that there exists a solution of (1.1) with the initial data satisfying (1.10), which can't be extended to larger strip than $S_{T(\Lambda)}$. The case (II) contains the heat equation, but our proof is completely different form that of **[1]** and **[12]**. In fact, the proof doesn't use the fundamental solution of the heat equation, and it depends only on the structure conditions of the equation. So our proof is applicable to more general equation than that of **[1]**.

For the proof of the case (II), we essentially use the techniques given in **[3]**–**[6]**. But it seems difficult to apply them to case (II) directly. To overcome this difficulty, we introduce a new weight function $\phi_\Lambda$ (see (2.8)), and obtain an $L^1(\mathbf{R}^N)$-estimate of $\phi_\Lambda$ (see (2.10)). From the estimate of $\|\phi_\Lambda(\cdot,t)\|_{L^1(\mathbf{R}^N)}$, we can estimate $\|u(\cdot,t)\|_{L^\infty(B_\rho)}$ and $\|\nabla u(\cdot,t)\|_{L^{p-1}(B_\rho)}$, and prove that the critical growth order of the initial data for this case is of exponential growth.

For the case of (I), there exists a weak solution of (1.1) under the initial data satisfying

$$\limsup_{\rho\to\infty} \rho^{-N-p/d} \int_{B_\rho} d|\mu| < \infty, \tag{1.11}$$

where $d = (p-1)/\beta - 1$. Furthermore for the case of (III), such that $Nd + p > 0$, we will give the $L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$-estimate of the solution for the $L^1_{\mathrm{loc}}(\mathbf{R}^N)$ initial data. For the cases of (I) and (III), our proof heavily depend an approach of **[3]**–**[6]**. Recently, for the case of (III) of the equation (1.1), V.Vespri **[13]** proved several inequalities, by which the existence of solutions is proved.

Finally, we remark that, to my acknowledgment, there is no results for the uniqueness of weak solutions of (1.1), though the uniqueness of strong solutions of (1.1) is given in **[11]**.

## 2. The Main Results

In this section, we give the definition of weak solution of (1.1) with (1.2), and state our results.

**Definition 1.** *A measurable function $u(x,t)$ defined in $\mathbf{R}^N \times (0,T)$ is a weak solution of (1.1) and (1.2), if for any $\epsilon \in (0,T)$ and every bounded open set $\Omega \subset \mathbf{R}^N$, $u \in L^{p-1}(0,T_\epsilon; W^{1,p-1}(\Omega))$, $|u|^{\beta-1}u \in C(0,T_\epsilon; L^1(\Omega))$ and*

$$\int_\Omega |u|^{\beta-1}u\varphi(x,t)dx + \int_0^t \int_\Omega \{-|u|^{\beta-1}u\varphi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\}dxd\tau = \int_\Omega \varphi(x,0)d\mu, \tag{2.1}$$

*for all $0 < t < T_\epsilon$ and all testing functions*

$$\varphi \in W^{1,\infty}(0,T_\epsilon; L^\infty(\Omega)) \cap L^\infty(0,T_\epsilon; W_0^{1,\infty}(\Omega)). \tag{2.2}$$

*Here $T_\epsilon = T - \epsilon$.*

Throughout this paper, we set

$$m = 1/\beta, \qquad d = (p-1)/\beta - 1 = m(p-1) - 1, \qquad \kappa_r = Nd + rp.$$

In particular, we set

$$\kappa = \kappa_1 = Nd + p, \qquad \kappa^* = \kappa_{mp} = Nd + mp^2$$

for simplicity. Furthermore by $C = C(A_1, A_2, \cdots)$ we denote positive constant which depends only $\beta, p, A_1, A_2, \cdots$.

**Case I: The Degenerate Case** $(d > 0)$

In order to represent the growth order of the initial data $\mu(x)$, we define $|||f|||_r$ by

$$|||f|||_r = \sup_{\rho \geq r} \rho^{-\kappa/d} \int_{B_\rho} |f|dx, \tag{2.3}$$

for $f \in L^1_{\text{loc}}(\mathbf{R}^N)$. This norm is a modification of the norm introduced in **[6]**.

**Theorem 1.** *Let $d > 0$ and $\mu$ be a $\sigma$-finite Borel measure in $\mathbf{R}^N$ satisfying*

$$|||\mu|||_r < \infty \qquad \textit{for some} \quad r > 0.$$

*Then there exists a weak solution $u$ of (1.1) and (1.2) in the strip $S_{T(\mu)}$, where*

$$T(\mu) = \begin{cases} C_1[\lim_{r\to\infty} |||\mu|||_r]^{-d} & \textit{if} \ \ \lim_{r\to\infty} |||\mu|||_r > 0 \\ +\infty & \textit{if} \ \ \lim_{r\to\infty} |||\mu|||_r = 0 \end{cases} \tag{2.4}$$

*and $C_1 = C_1(N, \beta, p)$.*

*Let $T_r(\mu) = C_1|||\mu|||_r^{-d}$. Then for any $t \in (0, T_r(\mu))$ and $\rho > 0$,*

$$|||\, |u|^\beta(\cdot, t)\, |||_r \le C_2|||\mu|||_r, \tag{2.5}$$

$$\|u(\cdot, t)\|_{L^\infty(B_\rho)} \le C_3 t^{-N/\beta\kappa} \rho^{p/\beta d} |||\mu|||_r^{p/\beta\kappa}, \tag{2.6}$$

*and*

$$\int_0^t \int_{B_\rho} |\nabla u|^{p-1} dx d\tau \le C_4 t^{1/\kappa} \rho^{1+\kappa/d} |||\mu|||_r^{1+d/\kappa}, \tag{2.7}$$

*where $C_i = C_i(N, \beta, p)$, $i = 2, 3, 4$.*

**Case II: The Critical Case** ($d = 0$)

For any $\lambda > 0$ and $\delta > 0$, let $\phi_\lambda(t)$ be a function defined by

$$\phi_\lambda(t) = \sup_{\tau \in (0,t)} \int_{\mathbf{R}^N} \eta(|x|) F(e^{g_\lambda} u^{p-1}) dx, \tag{2.8}$$

where

$$F(s) = \begin{cases} |s|^{1+\delta}/(1+\delta), & \text{if} \ \ |s| \le 1, \\ |s| - \delta/(1+\delta), & \text{if} \ \ |s| \ge 1, \end{cases} \qquad \eta(s) = \begin{cases} 1, & \text{if} \quad s \le 1, \\ s^{(N+p)/(p-1)}, & \text{if} \quad s \ge 1, \end{cases}$$

and

$$g_\lambda(x, t) = -\lambda \left( \frac{|x|^p}{1-t} \right)^{1/(p-1)} (1 + t^l), \qquad 0 < l < 1/2. \tag{2.9}$$

Then our result for the case of (II) is as follows.

**Theorem 2.** *Let $d = 0$ and $\mu$ be a $\sigma$-finite Borel measure satisfying*

$$\int_{\mathbf{R}^N} \exp(-\Lambda|x|^{p/(p-1)})d|\mu(x)| < \infty \tag{2.10}$$

*for some $\Lambda > 0$. Then there exists a weak solution $u$ of (1.1) and (1.2) in the strip $S_{T(\Lambda)}$, where $T(\Lambda)$ is a constant such that*

$$T(\Lambda) = \frac{(p-1)^{2(p-1)}}{p^p}\Lambda^{1-p}. \tag{2.11}$$

*Furthermore let $\delta > 0$ be a sufficiently small constant. Then for any $\lambda > \Lambda$ there exists a constant $T_0 < 1$ such that $u$ satisfies the following inequalities,*

$$\phi_\lambda(t) \le C_1\phi_\lambda(0) \le C\int_{\mathbf{R}^N} \exp(-\Lambda|x|^{p/(p-1)})d|\mu(x)| \tag{2.12}$$

$$\|e^{g_\lambda(\cdot,t)}u^{p-1}\|_{L^\infty(\mathbf{R}^N)} \le C_2(1+t^{-N/p})\phi_\lambda(t) \tag{2.13}$$

*for all $t \in (0, T_0)$, where $C = C(p, N, \Lambda, \lambda, \delta)$ and $C_i = C_i(p, N, \delta)$, $i = 1, 2$.*

The estimate of $T(\Lambda)$ in (2.11) is optimal in the sense that there exists a solution blowing up at $T(\Lambda)$. In fact, let $u(x,t)$ be a function such that

$$u(x,t) = (1-\sigma t)^{-N/p(p-1)}\exp\left[\gamma\left(\frac{|x|^p}{1-\sigma t}\right)^{1/(p-1)}\right], \qquad \gamma = \frac{p-1}{p}\left(\frac{\sigma}{p}\right)^{1/(p-1)},$$

where $\sigma$ is any positive constant. Then $u(x,t)$ is a solution of (1.1) in the strip $S_{1/\sigma}$, and

$$u^{p-1}(x,0) = \exp((p-1)\gamma|x|^{p/(p-1)}).$$

Then $T((p-1)\Lambda) = 1/\sigma$ and $u(x,t)$ blows up at $t = 1/\sigma$.

**Case III: The Singular Case** ($d < 0$)

We only treat the result for the case of $\kappa > 0$.

**Theorem 3.** *Let $d < 0$ be a positive constant such that $\kappa > 0$ and let $\mu$ be a $\sigma$-finite Borel measure in $\mathbf{R}^N$. Then there exists a weak solution $u$ of (1.1) and (1.2) in $\mathbf{R}^N \times (0,\infty)$. Furthermore the solution $u(x,t)$ satisfies the following inequalities,*

$$\|\,|u|^\beta(\cdot,t)\,\|_{L^\infty(B_\rho)} \le C_1 t^{-N/\kappa}\left(\sup_{0<\tau<t}\int_{B_{2\rho}} |u|^\beta(x,\tau)dx\right)^{p/\kappa} + C_2\left(\frac{t}{\rho^p}\right)^{p/\kappa}$$

*and*

$$\sup_{0<\tau<t}\int_{B_\rho}|u|^\beta(x,\tau)dx \le C_3\int_{B_{2\rho}} d|\mu(x)| + C_4\left(\frac{t}{\rho^\kappa}\right)^{1/-d},$$

*for any $t > 0$ and $\rho > 0$, where $C_i = C_i(N, m, p)$, $i = 1, 2, 3, 4$.*

The essential part of Theorem 3 was proved by V.Vespri. See Theorem 2–1 and Theorem 2–2 in [**13**].

We remark that the estimates of solutions given in Theorem 1–Theorem 3 may be extended to nonnegative strong subsolutions of

$$\frac{\partial}{\partial t}(|u|^{\beta-1}u) - \mathrm{div}A(x,t,u,\nabla u) \le B(x,t,u,\nabla u). \tag{2.14}$$

Here the structure conditions given below are satisfied:

$$\begin{cases} C_1|q|^p - g_1(x,t) \le A(x,t,u,q)\cdot q \le C_2|q|^p + g_2(x,t), \\ [A(x,t,u,q) - A(x,t,u,\tilde{q})]\cdot(q-\tilde{q}) \ge 0, \\ |B(x,t,u,q)| \le C_3|q|^{p-1} + g_3(x,t) \end{cases}$$

for any $(x,t,u) \in \mathbf{R}^N \times \mathbf{R}_+ \times \mathbf{R}$ and $q, \tilde{q} \in \mathbf{R}^N$, where $C_i$, $i = 1, 2, 3$, are given constants and $g_i$, $i = 1, 2$ are given bounded functions in $\mathbf{R}^{N+1}$.

Furthermore, from Lemma 1–2 in [**3**], $u_+ = \max\{u, 0\}$ and $u_- = -\min\{u, 0\}$ are nonnegative weak subsolutions of (1.1), and the estimates of Theorem 1–Theorem 3 holds for nonnegative weak subsolutions of (2.14). Therefore throughout this paper, we treat only nonnegative solutions of (1.1).

In the following section, we give the proof of Theorem 2 for the heat equation. The proof for the heat equation contains an essential part of Theorem 2 for the equation (1.1). For the cases (I) and (III), we may complete the proofs of Theorem 1 and Theorem 3 by modifying the proof of the results of [**3**]–[**6**].

■

## 3. Proof of Theorem 2 for the Heat Equation

In this section, we consider the Cauchy problem of the heat equation,

$$\begin{cases} u_t = \Delta u & \text{in} \quad \mathbf{R}^N \times (0,T), \\ u(x,0) = \mu(x) \ge 0 & \text{in} \quad \mathbf{R}^N, \end{cases} \tag{3.1}$$

where $\mu \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ satisfying the condition (2.10). In order to prove Theorem 2 for the heat equation, we need the following two propositions.

**Proposition 3–1.** *(See [1]) Let $\mu$ be a nonnegative $L^2_{\mathrm{loc}}(\mathbf{R}^N)$-function such that*

$$\mu(x)\exp(-\Lambda|x|^2) \in L^2(\mathbf{R}^N). \tag{3.2}$$

*Then there exists a weak solution $u(x,t)$ of (3.1) in $S_{T(\Lambda)}$, where $T(\Lambda)$ is a constant given in (1.11). Furthermore for any $\epsilon \in (0, T(\Lambda))$ there exist constants $C_1^\epsilon$ and $C_2^\epsilon$ such that the solution $u(x,t)$ satisfies*

$$\sup_{0<\tau<T_\epsilon} \int_{\mathbf{R}^N} \exp(-C_1^\epsilon|x|^2)u^2(x,\tau)dx + \iint_{S_{T_\epsilon}} \exp(-C_1^\epsilon|x|^2)|\nabla u|^2 dxd\tau \le C_2^\epsilon \, \|\mu \exp(-\Lambda|\cdot|^2)\|_{L^2(\mathbf{R}^N)}, \tag{3.3}$$

*where $T_\epsilon = T(\Lambda) - \epsilon$.*

The following proposition is proved by the arguments similar to those of [3]–[6].

**Proposition 3–2.** *Let $\eta$ and $g_\lambda$ be functions appearing in (2.8) and $r$ be a constant with $r > 1$. For any $\lambda > 0$, there exist constants $C = C(r, \lambda)$ and $T_\lambda^* < 1$ such that*

$$\|\eta(|x|)e^{g_\lambda}u\|_{L^\infty(B_{R_1}\times(t_1,t))} \le CM^{(N+2)/2r}\left(\int_{t_2}^{t}\int_{B_{R_2}} [\eta(|x|)e^{g_\lambda}u]^r dxd\tau\right)^{1/r}, \tag{3.4}$$

*for any $R_1$, $R_2$ with $0 < R_1 < R_2$ and any $t, t_1, t_2$ with $0 < t_2 < t_1 < t \le T_\lambda^*$, where*

$$M = |\nabla g_\lambda|^2(R_2, t) + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}. \tag{3.5}$$

Now we begin with proving Theorem 2 for the heat equation. From Proposition 3–2, we have the following proposition.

**Proposition 3–3.** *Let $u$ be a solution of (3.1). Then for $0 < T_0 < T_\lambda^*$, there exists a constant $C = C(N, \delta, T_0)$ such that*

$$\|e^{g_\lambda}u\|_{L^\infty(B_\rho)}(t) \le C + Ct^{-N/2}\phi_\lambda(t) \tag{3.6}$$

*for $0 < t < T_0$.*

**Proof of Proposition 3–3:** For any $\rho > 0$ and $t > 0$, let $Q_s$ be a cylinder defined by $B_{\rho_s} \times (t_s, t)$, where $\rho_s = \sum_{i=1}^{s}(2^{-i-1})\rho$ and $t_s = (1 - \sum_i^s(2^{-i-1}))t$. Then applying Proposition 3–2 to the pair of cylinders $Q_s \subset Q_{s+1}$, we have

$$\|\eta e^{g_\lambda}u\|_{L^\infty(Q_s)} \le C\frac{M^{(N+2)/2(1+\delta)}}{b^{s/(1+\delta)}}\left(\iint_{Q_{s+1}} [\eta e^{g_\lambda}u]^{1+\delta}dxd\tau\right)^{1/(1+\delta)}, \tag{3.7}$$

where $M = |\nabla g_\lambda|^2(2\rho, t) + \rho^{-2} + t^{-1}$ and $b = 2^{-N-2}$.

For any measurable set $E$ in $\mathbf{R}^N$, we denote by $\chi_E$ the characteristic function of $E$, and set

$$\chi_1(x) = \chi_{\{e^{g_\lambda} u \le 1\}}(x), \qquad \chi_2(x) = \chi_{\{e^{g_\lambda} u \ge 1\}}(x). \tag{3.8}$$

Then from (2.8) we have

$$\begin{aligned}\iint_{Q_{s+1}} [\eta(|x|)e^{g_\lambda} u]^{1+\delta} dx d\tau &\le \eta^\delta(2\rho) \iint_{Q_{s+1}} \eta(|x|)[e^{g_\lambda} u]^{1+\delta} \chi_1(x) dx d\tau \\ &\quad + \|\eta e^{g_\lambda} u\|^\delta_{L^\infty(Q_{s+1})} \iint_{Q_{s+1}} \eta(|x|) e^{g_\lambda} u dx d\tau \\ &\le C[\eta^\delta(2\rho) + \|\eta e^{g_\lambda} u\|^\delta_{L^\infty(Q_{s+1})}] \int_0^t \phi_\lambda(\tau) d\tau.\end{aligned}$$

So combining (3.7), we have

$$\begin{aligned}\|\eta e^{g_\lambda} u\|_{\infty, Q_s} &\le C \frac{M^{(N+2)/2(1+\delta)}}{b^{s/(1+\delta)}} \\ &\quad \times \left( [\eta^\delta(2\rho) + \|\eta e^{g_\lambda} u\|^\delta_{L^\infty(Q_{s+1})}] \int_0^t \phi_\lambda(\tau) d\tau \right)^{1/(1+\delta)}.\end{aligned}$$

Then from the Young inequality, for any $\nu > 0$ there exists a constant $C = C(\nu, \delta)$ such that

$$\begin{aligned}\|\eta e^{g_\lambda} u\|_{L^\infty(Q_s)} &\le \nu \|\eta e^{g_\lambda} u\|_{L^\infty(Q_{s+1})} \\ &\quad + C(\nu, \delta) 2^{(N+2)s} [\eta(2\rho) + M^{(N+2)/2} \int_0^t \phi_\lambda(\tau) d\tau].\end{aligned}$$

Therefore iteration of these inequalities yields

$$\begin{aligned}\|\eta e^{g_\lambda} u\|_{L^\infty(Q_0)} &\le \nu^s \|\eta e^{g_\lambda} u\|_{L^\infty(Q_s)} \\ &\quad + C(\nu, \delta)[\eta(2\rho) + M^{(N+2)/2} t \phi_\lambda(t)] \sum_{i=1}^s (\nu 2^{N+2})^i.\end{aligned}$$

Therefore we set $\nu = 2^{-(N+3)}$, take the limit as $s \to \infty$, and obtain

$$\|\eta e^{g_\lambda} u\|_{L^\infty(B_\rho)} \le C(\delta)\eta(2\rho) + C(\delta)[|\nabla g_\lambda|^2 + t^{-1}]^{(N+2)/2} t \phi_\lambda(t). \tag{3.9}$$

Therefore from (3.9), we have the inequality (3.6) for the case of $\rho \le 1$.

For any $0 < T_0 < 1$ there exists a constant $C = C(T_0)$ such that $|\nabla g_\lambda|^2(x, t) \le C|x|^2$. For the case of $\rho \ge 1$, from (3.9) and the definition of $\eta$ we obtain

$$\|e^{g_\lambda} u^{p-1}\|_{L^\infty(B_\rho \setminus B_{\rho/2})} \le C + C t^{-N/2} \phi_\lambda(t),$$

where $C = C(N, \delta, T_0)$. Therefore we obtain the inequality (3.6) for the case of $\rho \ge 1$. ∎

**Proposition 3–4.** *Let $\zeta(x)$ is a piecewise smooth cutoff function such that $\zeta \equiv 1$ on $B_\rho$, $|\nabla\zeta| \le 1/\rho$ and $\operatorname{supp}\zeta \subset B_{2\rho}$. Then there exists a constant $T_0 < 1$ dependent only on $p$ such that*

$$\limsup_{\epsilon\to 0} \int_0^t \int_{B_{2\rho}} \tau^{1/2}\eta(|x|)e^{g_\lambda}\frac{|\nabla u|^2}{u+\epsilon}[e^{g_\lambda}(u+\epsilon)]^\delta \zeta^2(x)\,dx\,d\tau \tag{3.10}$$
$$\le C\left[\int_0^t \tau^{-1/2}\phi_\lambda(\tau)d\tau + \int_0^t \tau^{\sigma-1}\phi_\lambda^{1+\delta}(\tau)d\tau\right],$$

*for $0 < t < T_0$ and $\rho \ge 1$, where $C = C(N,\delta)$ and $\sigma = (1-\delta N)/2$.*

**Proof of Proposition 3–4:** Let

$$\varphi_\epsilon(x,t) = t^{1/2}\eta(|x|)e^{g_\lambda}[e^{g_\lambda}(u+\epsilon)]^\delta\zeta^2(x).$$

Then we have

$$\liminf_{\epsilon\to 0}\int_0^t\int_{B_{2\rho}} u_t\varphi_\epsilon\,dx\,d\tau \ge -\frac{1}{2(1+\delta)}\int_0^t\int_{B_{2\rho}}\tau^{-1/2}\eta[e^{g_\lambda}u]^{1+\delta}dx\,d\tau$$
$$-\frac{1}{1+\delta}\int_0^t\int_{B_{2\rho}}\tau^{1/2}\eta[e^{g_\lambda}u]^{1+\delta}\frac{\partial}{\partial t}g_\lambda\,dx\,d\tau,$$

and

$$\int_0^t\int_{B_{2\rho}}\nabla u\cdot\nabla\varphi_\epsilon\,dx\,d\tau \ge \frac{\delta}{2}\int_0^t\int_{B_{2\rho}}\tau^{1/2}e^{g_\lambda}\frac{|\nabla u|^2}{u+\epsilon}[e^{g_\lambda}(u+\epsilon)]^\delta dx\,d\tau$$
$$-C(\delta)\int_0^t\int_{B_{2\rho}}\tau^{1/2}\eta[e^{g_\lambda}(u+\epsilon)]^{1+\delta}|\nabla g_\lambda|^2dx\,d\tau$$
$$-\frac{C(\delta)}{\rho^2}\int_0^t\int_{B_{2\rho}}\tau^{1/2}\eta[e^{g_\lambda}(u+\epsilon)]^{1+\delta}dx\,d\tau.$$

Taking a sufficiently small $T_0$, we obtain

$$\limsup_{\epsilon\to 0}\int_0^t\int_{B_{2\rho}}\tau^{1/2}\eta(|x|)e^{g_\lambda}\frac{|\nabla u|^2}{u+\epsilon}[e^{g_\lambda}(u+\epsilon)]^\delta\zeta^2dx\,d\tau \tag{3.11}$$
$$\le C\int_0^t\int_{B_{2\rho}}\tau^{-1/2}\eta(|x|)(e^{g_\lambda}u)^{\delta+1}dx\,d\tau.$$

Here we used the relation that $t^{1/2}/\rho^2 \le t^{-1/2}$ for $0 < t < T_0$. So we have

$$\int_0^t\int_{B_{2\rho}}\tau^{-1/2}\eta(|x|)(e^{g_\lambda}u)^{\delta+1}dx\,d\tau \le \int_0^t\tau^{-1/2}\phi_\lambda(\tau)d\tau \tag{3.12}$$
$$+\int_0^t\int_{B_{2\rho}}\tau^{-1/2}\eta(|x|)e^{g_\lambda}u\|e^{g_\lambda}u\|^\delta_{L^\infty(B_{2\rho})}(\tau)\chi_2\,dx\,d\tau.$$

Furthermore from Proposition 3–3 we have

$$
\begin{aligned}
(3.13)\qquad & \int_0^t \int_{B_{2\rho}} \tau^{-1/2}\eta(|x|)e^{g_\lambda} u\chi_2 \|e^{g_\lambda} u\|^{\delta}_{L^\infty(B_{2\rho})}(\tau)\,dx\,d\tau \\
& \le \int_0^t \int_{B_{2\rho}} \tau^{-1/2}\eta(|x|)e^{g_\lambda} u\chi_2 [1+\tau^{-\delta N/2}\phi^{\delta}_{\lambda}(\tau)]\,dx\,d\tau \\
& \le C\int_0^t \tau^{-1/2}\phi_\lambda(\tau)\,d\tau + C\int_0^t \tau^{-1/2-\delta N/2}\phi_\lambda^{1+\delta}(\tau)\,d\tau
\end{aligned}
$$

for any $0 < t < T_0$. By combining (3.11)–(3.12), we obtain the inequality (3.10). ∎

**Proposition 3–5.** *There exists a constant $T_0 = T_0(\delta, l, \phi_\lambda(0))$ such that*

$$
(3.14)\qquad \phi_\lambda(t) \le C\phi_\lambda(0)
$$

*for $0 < t < T_0$, where $C = C(\delta)$. Furthermore*

$$
(3.15)\qquad \|e^{g_\lambda(\cdot,t)}u\|_{L^\infty(B_\rho)} \le C(1+t^{-N/2})\phi_\lambda(0)
$$

*for any $\rho > 0$ and $0 < t < T_0$.*

**Proof of Proposition 3–5:** We set

$$
\varphi_\epsilon(x,t) = \eta(|x|)e^{g_\lambda}F'(e^{g_\lambda}(u+\epsilon))\zeta^2(x),
$$

where $\zeta$ is a function given in Proposition 3–4, and we have

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_0^t\int_{B_{2\rho}} u_t\varphi_\epsilon\,dx\,d\tau = & \int_{B_{2\rho}} \eta(|x|)F(e^{g_\lambda}u)\zeta^2\,dx\Big|_{\tau=0}^{\tau=t} \\
& - \int_0^t\int_{B_{2\rho}} \eta(|x|)[(e^{g_\lambda}u)^{1+\delta}\chi_1 + e^{g_\lambda}u\chi_2]\zeta^2\frac{\partial}{\partial t}g_\lambda\,dx\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t\int_{B_{2\rho}} \nabla u\cdot\nabla\varphi_\epsilon\,dx\,d\tau \ge\ & \delta\int_0^t\int_{B_{2\rho}} \eta e^{g_\lambda}\frac{|\nabla u|^2}{u+\epsilon}[e^{g_\lambda}(u+\epsilon)]^\delta\zeta^2\chi_1\,dx\,d\tau \\
& - C(\delta)\int_0^t\int_{B_{2\rho}} \eta e^{g_\lambda}|\nabla u|(|\nabla g_\lambda|\zeta^2 + \zeta|\nabla\zeta|)[e^{g_\lambda}(u+\epsilon)]^\delta\chi_1\,dx\,d\tau \\
& - \int_0^t\int_{B_{2\rho}} \eta e^{g_\lambda}|\nabla u|(|\nabla g_\lambda|\zeta^2 + \zeta|\nabla\zeta|)\chi_2\,dx\,d\tau,
\end{aligned}
$$

where $\chi_i$, $i = 1,2$ is given in (3.8). The Young inequality yields

$$\begin{aligned}\int_0^t \int_{B_{2\rho}} & \eta e^{g_\lambda} |\nabla u||\nabla g_\lambda| \zeta^2 [e^{g_\lambda}(u+\epsilon)]^\delta \chi_1 dx d\tau \\ &\le \int_0^t \int_{B_{2\rho}} \tau^{1/2} \eta e^{g_\lambda} \frac{|\nabla u|^2}{u+\epsilon} [e^{g_\lambda}(u+\epsilon)]^\delta \zeta^2 \chi_1 dx d\tau \\ &\qquad + \int_0^t \int_{B_{2\rho}} \eta [e^{g_\lambda}(u+\epsilon)]^{1+\delta} \tau^{-1/2} |\nabla g_\lambda|^2 \zeta^2 \chi_1 dx d\tau.\end{aligned}$$

and

$$\begin{aligned}\int_0^t \int_{B_{2\rho}} \eta e^{g_\lambda} |\nabla u||\nabla g_\lambda| \zeta^2 \chi_2 dx d\tau &\le \int_0^t \int_{B_{2\rho}} \tau^{1/2} \eta e^{g_\lambda} \frac{|\nabla u|^2}{u+\epsilon} \zeta^2 \chi_2 dx d\tau \\ &\quad + \int_0^t \int_{B_{2\rho}} \eta e^{g_\lambda} (u+\epsilon) \tau^{-1/2} |\nabla g_\lambda|^2 \zeta^2 \chi_2 dx d\tau.\end{aligned}$$

Therefore from (2.9) and the Young inequality, there exists a constant $C = C(N, \delta, l)$ such that

$$\begin{aligned}\int_{B_\rho} \eta(|x|) F(e^{g_\lambda} u) dx \Big|_{\tau=t} &\le \int_{B_{2\rho}} \eta(|x|) F(e^{g_\lambda} u) dx \Big|_{\tau=0} \\ &+ \limsup_{\epsilon\to 0} \int_0^t \int_{B_{2\rho}} \tau^{1/2} \eta(|x|) e^{g_\lambda} \frac{|\nabla u|^2}{u+\epsilon} [e^{g_\lambda}(u+\epsilon)]^\delta \zeta^2 dx d\tau \\ &+ \frac{C}{\rho^2} \int_0^t \int_{B_{2\rho}} \eta(|x|) (e^{g_\lambda} u)^{\delta+1} \chi_1 dx d\tau \\ &+ \frac{C}{\rho^2} \int_0^t \int_{B_{2\rho}} \tau^{-1/2} \eta(|x|) e^{g_\lambda} u \chi_2 dx d\tau.\end{aligned} \tag{3.16}$$

Therefore taking the limit $\rho \to \infty$, from Proposition 3–4, we have

$$\begin{aligned}\phi_\lambda(t) &\le \phi_\lambda(0) + C \int_0^t [(\tau^{-1/2} + \tau^{-1/2}) \phi_\lambda(\tau) + \tau^{\sigma-1} \phi_\lambda^{1+\delta}(\tau)] d\tau \\ &\le \phi_\lambda(0) + C(t^{1/2} + t^{1/2}) \phi_\lambda(t) + C \int_0^t \tau^{\sigma-1} \phi_\lambda^{1+\delta}(\tau) d\tau,\end{aligned}$$

where $\sigma$ is a constant given in Proposition 3–4. Here we take a sufficiently small $\delta > 0$ such that $\sigma > 0$, and fix $\delta$. Taking a sufficiently small $t$ such that $Ct^{1/2} \le 1/4$, we have

$$\phi_\lambda(t) \le 2\phi_\lambda(0) + C \int_0^t \tau^{\sigma-1} \phi_\lambda^{1+\delta}(\tau) d\tau. \tag{3.17}$$

It follows from (3.17) that $\phi_\lambda(t)$ is majorized by the solution of

$$H'(t) = Ct^{\sigma-1}H^{1+\delta}(t), \qquad H(0) = 2\phi_\lambda(0),$$

and so we obtain

$$\phi_\lambda(t) \le H(t) = 2\left[1 - \frac{C\delta 2^\delta}{1-\sigma} t^{-\sigma+1}\phi_\lambda^\delta(0)\right]^{-1/\delta} \phi_\lambda(0),$$

provided the bracket is positive for any $0 < t < T_0$. Therefore taking sufficiently small $T_0 > 0$ such that $[\cdots] > 0$, we complete the proof of Proposition 3–5. ∎

**Proposition 3–6.** *Under the assumption of Proposition 3–5, there holds*

$$\int_0^t \int_{B_\rho} \eta(|x|)e^{g_\lambda}|\nabla u| dx d\tau \le Ct^\sigma(\rho^N + \phi_\lambda(0)) \tag{3.18}$$

*for any* $0 < t < T_0$.

**Proof of Proposition 3–6:** By the Young inequality, we have

$$\begin{aligned}\int_0^t \int_{B_\rho} \eta(|x|)e^{g_\lambda}|\nabla u| dx d\tau \le & \int_0^t \int_{B_\rho} \tau^{1/2} e^{g_\lambda} \frac{|\nabla u|^2}{u+\epsilon}[e^{g_\lambda}(u+\epsilon)]^\delta dx d\tau \\ & + \int_0^t \int_{B_\rho} \tau^{-1/2}[e^{g_\lambda}(u+\epsilon)]^{1-\delta} dx d\tau \\ & \equiv I_1(\epsilon) + I_2(\epsilon).\end{aligned}$$

From Propositions 3–4 and 3–5, we have $\limsup_{\epsilon\to 0} I_1(\epsilon) \le Ct^\sigma\phi_\lambda(0)$. For the second term $I_2(\epsilon)$, we have

$$\begin{aligned}\lim_{\epsilon\to 0} I_2(\epsilon) \le & \int_0^t \int_{B_\rho} \tau^{-1/2}(e^{g_\lambda}u)^{1-\delta}[\chi_1 + \chi_2] dx d\tau \\ \le & Ct^{1/2}(\rho^N + \phi_\lambda(t)),\end{aligned}$$

and so we obtain (3.18). ∎

Now we complete the proof of Theorem 2 for the heat equation.

**Proof of Theorem 2:** Let $\lambda$ be any constant such that $\lambda > \Lambda$. From the definition of $\phi_\lambda$, we have

$$\phi_\lambda(0) \le \int_{\mathbf{R}^N} \eta(|x|)\exp(-\lambda|x|^2)\mu(x)dx.$$

Therefore from Proposition 4–3 and $\lambda > \Lambda$, we obtain the inequalities (2.12) and (2.13). for the initial data $\mu \in C_0^\infty(\mathbf{R}^N)$.

For any $\mu \in L^1_{\rm loc}(\mathbf{R}^N)$ satisfying (2.10), let $\mu_n$ be a function in $C_0^\infty(\mathbf{R}^N)$ such that

$$\lim_{n\to\infty} \int_{\mathbf{R}^N} \mu_n \exp(-\lambda|x|^2)dx = \int_{\mathbf{R}^N} \exp(-\lambda|x|^2)d\mu(x).$$

From Proposition 3–1, there exists a solution $u_n$ of (3.1) for $u(x,0) = \mu_n(x)$ in $S_\infty \equiv \mathbf{R}^N \times (0,\infty)$. Furthermore Proposition 3–1 and (3.15) yield

$$\int_\tau^{T_0} \int_{B_\rho} |\nabla u_n|^2 dxdt \le C(\rho,\tau)$$

for any $0 < \tau < T_0$ and $\rho \ge 1$, where $C(\rho,\tau)$ is a constant independent of $n$. If necessary, taking a subsequence of $\{u_n\}$, we see that

$$\lim_{n\to\infty} \int_\tau^{T_0} \int_{B_\rho} \nabla u_n \cdot \nabla\varphi dxdt = \int_\tau^{T_0} \int_{B_\rho} \nabla u \cdot \nabla\varphi dxdt$$

for any $\varphi \in L^\infty(0,T_0; W^{1,\infty}_{\rm loc}(\mathbf{R}^N))$. Therefore the function $u$ is a solution of (1.1) and (1.2) in $S_{T_0}$.

On the other hand, from (3.15) we have

$$\| \exp(-\lambda(t)|x|^2)u(x,t)\|_{L^\infty(\mathbf{R}^N)} < \infty$$

for $0 < t < T_0$, where $\lambda(t) = g_\lambda(0,t) = \lambda(1 + t^{1/2})$. Therefore from Proposition 3–1, there exists a solution of (1.1) and (1.2) in $S_{T(\Lambda)}$, and the proof of Theorem 2 for the heat equation is completed. ∎

## REFERENCES


1. D.G.Aronson, *Non-negative solutions of linear parabolic equations*, Annali. Scuola. Norm. Sup. Pisa Cl. Sci. **22** (1968), 607–694.
2. Ph.Benilan, M.G.Crandall and M.Pierre, *Solutions of the porous medium equation under optimal conditions on initial values*, Indiana Univ. Math. Jour. **33** (1984), 51–87.
3. E.Di Benedetto, "Degenerate Parabolic Equations," Springer-Verlag, 1993.
4. E.Di Benedetto and M.A.Herrero, *On the Cauchy problems and initial traces for a degenerate parabolic equation*, Trans. Amer. Math. Soc. **314** (1989), 187–224.

5. E.Di Benedetto and M.A.Herrero, *Nonnegative solutions of the evolution p-Laplacian equations. Initial traces and Cauchy problem when* $1 < p < 2$, Arch. Rational Mech. Anal. **111** (1990), 225–290.

6. E.Di Benedetto and T.C.Kwong, *Harnack estimates and extinction profile for weak solutions of certain singular parabolic equations*, Trans. Amer. Math. Soc. **330** (1992), 783–811.

7. J.L.Lions, "Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires," Dunod , Gauthier-Villars, Paris, 1969.

8. P.A.Raviart, *Sur la résolution de certains équations paraboliques nonlinéaires*, Jour. Func. Anal. **5** (1970), 299–328.

9. S.Tacklind, *Sur les classes quasianalitiques des solutions des équations aux derivée partielles du type parabolique*, Acta Reg. Soc. Sc. Uppsaliensis **10(3)** (1936), 3-55.

10. N.S.Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 206–226.

11. M.Tsutsumi, *On solutions of some doubly nonlinear degenerate parabolic equations*, J. Math. Anal. Appl. **132** (1988), 187–212.

12. A.N.Tychonov, *thórèmes d'unicité pour l'équation de la chaleur*, Mat. Sb. **42** (1935), 199-216.

13. V.Vespri, *Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations*, Jour. Math. Anal. Appl. **181** (1994), 104–131.

14. D.V.Widder, *Positive temperatures in an infinite rod*, Trans. A.M.S. **55** (1944), 85–95.